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CLASSIFICATION OF EXTINCTION PROFILES FOR A ONE-DIMENSIONAL DIFFUSIVE HAMILTON-JACOBI EQUATION WITH CRITICAL ABSORPTION

RAZVAN GABRIEL IAGAR AND PHILIPPE LAURENÇOT

ABSTRACT. A classification of the behavior of the solutions $f(\cdot, a)$ to the ordinary differential equation $(|f'|^{p-2}f')' + f - |f'|^{p-1} = 0$ in $(0, \infty)$ with initial condition $f(0, a) = a$ and $f'(0, a) = 0$ is provided, according to the value of the parameter $a > 0$, the exponent p ranging in $(1, 2)$. There is a threshold value a_* which separates different behaviors of $f(\cdot, a)$: if $a > a_*$ then $f(\cdot, a)$ vanishes at least once in $(0, \infty)$ and takes negative values while $f(\cdot, a)$ is positive in $(0, \infty)$ and decays algebraically to zero as $r \rightarrow \infty$ if $a \in (0, a_*)$. At the threshold value, $f(\cdot, a_*)$ is also positive in $(0, \infty)$ but decays exponentially fast to zero as $r \rightarrow \infty$. The proof of these results relies on a transformation to a first-order ordinary differential equation and a monotonicity property with respect to $a > 0$. This classification is one step in the description of the dynamics near the extinction time of a diffusive Hamilton-Jacobi equation with critical gradient absorption and fast diffusion.

1. INTRODUCTION

Let $p \in (1, 2)$. Owing to its scale invariance, the diffusive Hamilton-Jacobi equation

$$\partial_t u - \partial_x (|\partial_x u|^{p-2} \partial_x u) + |\partial_x u|^{p-1} = 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}, \quad (1.1)$$

is expected to have self-similar solutions with separate variables, that is, solutions of the form

$$u_s(t, x) = ((2-p)(T-t)_+)^{1/(2-p)} f(|x|), \quad (t, x) \in (0, \infty) \times \mathbb{R}, \quad (1.2)$$

which vanish identically after a finite time $T > 0$. Inserting this ansatz in (1.1) leads us to the ordinary differential equation

$$(|f'|^{p-2}f')' + f - |f'|^{p-1} = 0, \quad r \in (0, \infty), \quad (1.3)$$

along with the boundary condition $f'(0) = 0$ stemming from the assumed symmetry and the expected smoothness of u_s with respect to the space variable. It is then natural to investigate the behavior of solutions to (1.3) according to the initial value $f(0)$. The main motivation for such an analysis is that non-negative self-similar solutions of the form (1.2) are expected to provide an accurate description of the behavior near the extinction time of non-negative solutions to (1.1) which enjoy the finite time extinction property. Indeed, it follows from [4, Theorem 1.2] that there are many non-negative solutions to (1.1) satisfying the latter property. The classification of solutions to (1.3) performed

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below allows us to identify the behavior at the extinction time of non-negative solutions to (1.1) in the companion paper [7], the initial data being even in \mathbb{R} , non-increasing on $(0, \infty)$ and decaying sufficiently rapidly as $x \rightarrow \infty$.

More precisely the main result of this paper is the following classification:

Theorem 1.1. *Given $a > 0$ there is a unique solution $f(\cdot, a)$ to the initial value problem*

$$(|f'|^{p-2}f')' + f - |f'|^{p-1} = 0, \quad r \in (0, \infty), \quad (1.4)$$

$$f(0, a) = a, \quad f'(0, a) = 0, \quad (1.5)$$

and

$$R(a) := \inf \{r > 0 : f(r, a) = 0\} \in (0, \infty]. \quad (1.6)$$

Furthermore there is $a_* > 0$ with the following properties:

- (a) if $a > a_*$ then $R(a) < \infty$, $f(R(a), a) = 0$, and $f'(R(a), a) < 0$.
- (b) if $a = a_*$ then $R(a_*) = \infty$ and there is $\ell_* > 0$ such that

$$\lim_{r \rightarrow \infty} e^{r/(p-1)} f(r, a_*) = \ell_*.$$

- (c) if $a \in (0, a_*)$ then $R(a) = \infty$ and

$$\lim_{r \rightarrow \infty} r^{(2-p)/(p-1)} f(r, a) = \left(\frac{p-1}{2-p} \right)^{(2-p)/(p-1)}.$$

Before giving a rough account of the proof of Theorem 1.1, let us complete the discussion started before the statement of Theorem 1.1 on the role of self-similar solutions to (1.1) of the form (1.2) in the description of the dynamics of non-negative solutions to (1.1) near their extinction time. According to Theorem 1.1 we have infinitely many non-negative self-similar solutions of the form (1.2) (corresponding to $a \in (0, a_*]$), but it turns out that *only one is selected* by the dynamics of (1.1) as the behavior near the extinction time. More precisely, as shown in [7], if u is a solution to (1.1) emanating from a non-negative even initial condition which is non-increasing on $(0, \infty)$ and decays sufficiently rapidly as $x \rightarrow \infty$ and if T_e denotes its extinction time, then $u(t, x)$ behaves as $((2-p)(T_e - t)_+)^{1/(2-p)} f(|x|, a_*)$ as $t \rightarrow T_e$. Let us point out that this *universal* behavior is also true in higher space dimensions $N \geq 2$ for $p \in (2N/(N+1), 2)$, but the identification of the corresponding self-similar profile is more involved and requires completely different arguments [7]. We also point out that a similar dynamics as the one described above is observed for the fast diffusion equation

$$\partial_t v - \Delta v^m + v^m = 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}^N,$$

when $m \in ((N-2)_+/N, 1)$ [2, 3].

Let us now describe more precisely the proof of Theorem 1.1. Given $a > 0$, classical results guarantee the well-posedness of (1.4)-(1.5), see Section 2. In addition, there is $R(a) \in (0, \infty]$ such that $f(\cdot, a)$ is a decreasing one-to-one function from $[0, R(a))$ onto $(0, a]$. This property allows us to introduce $\psi(\cdot, a)$ defined on $(0, 1)$ by

$$\psi \left(1 - \frac{f(r, a)}{a}, a \right) := \frac{|f'(r, a)|^p}{a^p}, \quad r \in [0, R(a)). \quad (1.7)$$

Thanks to (1.4)-(1.5), the function $\psi(\cdot, a)$ solves

$$\begin{aligned} \psi'(y) + \frac{p}{p-1} \psi(y)^{(p-1)/p}(y) &= \frac{p}{p-1} a^{2-p}(1-y), \quad y \in (0, 1), \\ \psi(0) &= 0. \end{aligned} \quad (1.8)$$

The transformation (1.7) thus reduces the second-order differential equation (1.4) to the first-order differential equation (1.8), which is already a valuable feature, but it also has the very interesting property that $\psi(\cdot, a)$ is monotone with respect to a . The latter is in particular of utmost importance to investigate uniqueness issues, see [1, 5, 6, 10, 13] for instance, where monotonicity with respect to the shooting parameter is used to establish uniqueness of the “fast orbit” for related problems. In addition, the finiteness of $R(a)$ as well as the behavior of $f(r, a)$ as $r \rightarrow \infty$ when $R(a) = \infty$ are directly connected to the behavior of $\psi(y, a)$ as $y \rightarrow 1$. The core of the analysis is actually the identification of the behavior of $\psi(y, a)$ as $y \rightarrow 1$ according to the value of a and is performed in Section 3. Interpreting the results obtained in Section 3 in terms of $f(\cdot, a)$ is done in Section 4, where we prove Theorem 1.1.

We end this introduction with a couple of remarks: on the one hand, the approach developed in this paper does not seem to extend to the study and classification of self-similar solutions to (1.1) of the form (1.2) in several space dimensions, the main reason being that the variable $r = |x|$ remains in the equation satisfied by ψ . Indeed, it seems that no transformation similar to (1.7) is available in dimension $N \geq 2$. Still, it is possible to establish a result similar to Theorem 1.1 in higher space dimensions but completely different arguments are used [7]. On the other hand, there is a *striking difference* between (1.3) and

$$(|f'|^{p-2} f')' + f - |f|^{p-2} f = 0, \quad r \in (0, \infty), \quad (1.9)$$

which involves only zero order reaction terms. Indeed, in general, (1.9) and its generalizations have only one non-negative C^1 -smooth solution which is defined on $(0, \infty)$ and converges to zero as $r \rightarrow \infty$, the so-called *ground state solution*, see [8, 9, 11, 12] and the references therein. This is in sharp contrast with (1.3) for which infinitely many ground states exist, see Theorem 1.1, but a single one features a faster decay as $r \rightarrow \infty$. This multiplicity of course complicates the analysis, as it requires not only to identify the possible decay rates as $r \rightarrow \infty$, but also the corresponding ranges of the parameter a .

2. WELL-POSEDNESS OF (1.4)-(1.5)

We begin with the well-posedness of (1.4)-(1.5) and basic properties of its solutions.

Lemma 2.1. *Given $a > 0$, there is a unique solution $f(\cdot, a) \in C^1([0, \infty))$ to (1.4)-(1.5) such that $|f'|^{p-2} f' \in C^1([0, \infty))$. Furthermore,*

$$R(a) = \inf \{r > 0 : f(r, a) = 0\} \in (0, \infty]$$

and $f(\cdot, a)$ enjoys the following properties:

$$0 < f(r, a) < a \quad \text{and} \quad -\left(a(1 - e^{-r})\right)^{1/(p-1)} < f'(r, a) < 0, \quad r \in (0, R(a)), \quad (2.1)$$

and

$$\frac{d}{dr} (e^r |f'(r, a)|^{p-2} f'(r, a)) = -e^r f(r, a) , \quad r \in (0, R(a)) . \quad (2.2)$$

Proof. Since $p \in (1, 2)$, the Cauchy-Lipschitz theorem ensures the existence and uniqueness of a solution $(f, g) \in C^1([0, \mathcal{R}(a)]; \mathbb{R}^2)$ to the initial value problem

$$\begin{aligned} f'(r) &= -|g(r)|^{(2-p)/(p-1)} g(r) , & g'(r) &= -|g(r)| + f(r) , & r &\in (0, \mathcal{R}(a)) , \\ f(0) &= a , & g(0) &= 0 , \end{aligned} \quad (2.3)$$

where $\mathcal{R}(a) \in (0, \infty]$ is such that either $\mathcal{R}(a) = \infty$ or

$$\mathcal{R}(a) < \infty \quad \text{and} \quad \limsup_{r \rightarrow \mathcal{R}(a)} (|f(r)| + |g(r)|) = \infty . \quad (2.4)$$

Since $g(r) = -|f'(r)|^{p-2} f'(r)$ by (2.3) for $r \in [0, \mathcal{R}(a))$, it readily follows from (2.3) that f solves (1.4)-(1.5). A further consequence of (2.3) is that

$$\begin{aligned} \frac{d}{dr} \left[\frac{p-1}{p} |g|^{p/(p-1)} + \frac{1}{2} f^2 \right] &= |g|^{(2-p)/(p-1)} g(f - |g|) - |g|^{(2-p)/(p-1)} g f \\ &= -|g|^{1/(p-1)} g \\ &\leq \frac{p}{p-1} \left[\frac{p-1}{p} |g|^{p/(p-1)} + \frac{1}{2} f^2 \right] , \end{aligned}$$

which excludes the occurrence of (2.4). Therefore $\mathcal{R}(a) = \infty$ and the positivity of a along with the continuity of f guarantee that $R(a) > 0$.

We next infer from (1.4)-(1.5) that

$$\lim_{r \rightarrow 0} (|f'|^{p-2} f')'(r) = -a < 0 ,$$

which implies that f' is negative in a right neighborhood of $r = 0$ as $f'(0) = 0$. Using again (1.4) we note that

$$\frac{d}{dr} (e^r |f'(r)|^{p-2} f'(r)) = e^r [|f'(r)|^{p-2} f'(r) + |f'(r)|^{p-1} - f(r)] , \quad r > 0 . \quad (2.5)$$

Consequently, as long as $f'(r)$ is negative and $r \in (0, R(a))$, there holds

$$\frac{d}{dr} (e^r |f'(r)|^{p-2} f'(r)) = -e^r f(r) < 0 ,$$

from which we deduce that f' cannot vanish in $(0, R(a))$. We have thus proved that $f'(r) < 0$ and $f(r) \in (0, a)$ for $r \in (0, R(a))$ as well as (2.2). Combining these properties gives

$$-\frac{d}{dr} (e^r |f'(r)|^{p-1}) \geq -ae^r , \quad r \in (0, R(a)) ,$$

hence, after integration and using (1.5),

$$-e^r |f'(r)|^{p-1} \geq -a(e^r - 1) , \quad r \in (0, R(a)) .$$

This completes the proof of Lemma 2.1. □

3. AN ALTERNATIVE FORMULATION

Let $a > 0$ and set $f = f(\cdot, a)$. As $f' < 0$ in $(0, R(a))$ by (2.1), the function $a - f$ is an increasing one-to-one function from $[0, R(a))$ onto $[0, a)$ and we denote its inverse by F . Then F is an increasing function from $[0, a)$ onto $[0, R(a))$ and we may define

$$\psi(y) = \psi(y, a) := \frac{1}{a^p} |f'(F(ay))|^p, \quad y \in [0, 1). \quad (3.1)$$

Equivalently,

$$\psi \left(1 - \frac{f(r)}{a} \right) = \frac{|f'(r)|^p}{a^p}, \quad r \in [0, R(a)), \quad (3.2)$$

and

$$\psi' \left(1 - \frac{f(r)}{a} \right) = -\frac{p}{(p-1)a^{p-1}} (|f'|^{p-2} f')'(r), \quad r \in [0, R(a)). \quad (3.3)$$

We then infer from (1.4)-(1.5), (3.2), and (3.3) that ψ solves

$$\psi'(y) + \frac{p}{p-1} \psi(y)^{(p-1)/p} = \frac{pa^{2-p}}{p-1} (1-y), \quad y \in (0, 1), \quad (3.4)$$

$$\psi(0) = 0. \quad (3.5)$$

We also deduce from (3.4)-(3.5) that

$$\psi'(0) = \frac{pa^{2-p}}{p-1} > 0. \quad (3.6)$$

3.1. Comparison and monotonicity. Though the equation (3.4) involves the exponent $(p-1)/p$, which ranges in $(0, 1)$, the following comparison principle is available:

Lemma 3.1 (Comparison principle). *Let $\xi_i \in C^1([0, 1])$, $i = 1, 2$, be two functions satisfying $\xi_1(0) \leq \xi_2(0)$ and*

$$\xi_1'(y) + \frac{p}{p-1} \xi_1(y)^{(p-1)/p} \leq \xi_2'(y) + \frac{p}{p-1} \xi_2(y)^{(p-1)/p}, \quad y \in (0, 1). \quad (3.7)$$

Then $\xi_1(y) \leq \xi_2(y)$ for $y \in [0, 1]$.

Proof. Lemma 3.1 actually follows from the monotonicity of $z \mapsto z^{(p-1)/p}$ and we recall its proof for the sake of completeness. Let $\delta > 0$ and define

$$y_\delta := \inf \{y \in [0, 1) : \xi_1(y) = \xi_2(y) + \delta\}.$$

Clearly $y_\delta > 0$ since $\xi_1(0) - \xi_2(0) - \delta \leq -\delta < 0$. Assume for contradiction that $y_\delta < 1$. Then $\xi_1 - \xi_2 - \delta < 0$ in $[0, y_\delta)$ and $(\xi_1' - \xi_2')(y_\delta) \geq 0$, while (3.7) gives

$$\begin{aligned} (\xi_1' - \xi_2')(y_\delta) &\leq \frac{p}{p-1} \xi_2(y_\delta)^{(p-1)/p} - \frac{p}{p-1} \xi_1(y_\delta)^{(p-1)/p} \\ &= \frac{p}{p-1} \xi_2(y_\delta)^{(p-1)/p} - \frac{p}{p-1} (\xi_2(y_\delta) + \delta)^{(p-1)/p} < 0, \end{aligned}$$

and a contradiction. Consequently, $\xi_1 \leq \xi_2 + \delta$ in $[0, 1)$ and, since this inequality is valid for any $\delta > 0$, we conclude that $\xi_1 \leq \xi_2$ in $[0, 1)$. \square

The transformation (3.1) has thus reduced the second-order equation (1.4) to the first-order equation (3.4), which lowers the complexity of the problem. An additional property, which turns out to be of high interest as well, of solutions to (3.4)-(3.5) is their monotonicity with respect to a , which is obviously a simple consequence of the comparison principle established in Lemma 3.1. A more precise result is actually available.

Lemma 3.2 (Monotonicity with respect to a). *Consider $0 < a_1 < a_2$. Then there exists $K(p) > 0$ depending only on p such that, for $y \in [0, 1)$,*

$$\begin{aligned} \psi(y, a_1) &\leq \psi(y, a_2) \leq \psi(y, a_1) + K(p)(a_2 - a_1)^{2-p}, \\ |\psi'(y, a_1) - \psi'(y, a_2)| &\leq K(p) [(a_2 - a_1)^{(2-p)(p-1)/p} + (a_2 - a_1)^{2-p}]. \end{aligned}$$

In addition, $\psi(y, a_1) < \psi(y, a_2)$ for any $y \in (0, 1)$.

Proof. Set $\psi_i = \psi(\cdot, a_i)$, $i = 1, 2$. Since $a_1 < a_2$, it readily follows from (1.4)-(1.5) that we can apply Lemma 3.1 with $(\xi_1, \xi_2) = (\psi_1, \psi_2)$. Consequently, $\psi_1 \leq \psi_2$ in $[0, 1)$.

We next put $M := p(a_2^{2-p} - a_1^{2-p})/(p-1)$ and $\xi_2(y) = \psi_1(y) + My$ for $y \in [0, 1)$. Then $\xi_2(0) = 0 = \psi_2(0)$ and it follows from (3.4) that, for $y \in (0, 1)$,

$$\begin{aligned} \xi_2'(y) + \frac{p}{p-1}\xi_2(y)^{(p-1)/p} &\geq \psi_1'(y) + M + \frac{p}{p-1}\psi_1(y)^{(p-1)/p} \\ &\geq M(1-y) + \frac{pa_1^{2-p}}{p-1}(1-y) = \frac{pa_2^{2-p}}{p-1}(1-y) \\ &= \psi_2'(y) + \frac{p}{p-1}\psi_2(y)^{(p-1)/p}. \end{aligned}$$

Applying Lemma 3.1 to $(\xi_1, \xi_2) = (\psi_2, \xi_2)$ entails that $\psi_2 \leq \xi_2$ in $[0, 1)$, which completes the proof of the first statement of Lemma 3.2. We next infer from (3.4), the Hölder continuity of $z \mapsto z^{(p-1)/p}$, and the first statement of Lemma 3.2 that

$$\begin{aligned} |\psi_1'(y) - \psi_2'(y)| &\leq \frac{p}{p-1}|\psi_1(y) - \psi_2(y)|^{(p-1)/p} + \frac{p}{p-1}(a_2 - a_1)^{2-p} \\ &\leq \frac{p}{p-1}K(p)^{(p-1)/p}(a_2 - a_1)^{(2-p)(p-1)/p} + \frac{p}{p-1}(a_2 - a_1)^{2-p}, \end{aligned}$$

and thus complete the proof of the continuous dependence with respect to a .

Finally, since $a_1 < a_2$, it follows that

$$\bar{y} := \sup\{y \in (0, 1) : \psi_1(z) < \psi_2(z) \text{ for } z \in (0, y)\} > 0.$$

Assume for contradiction that $\bar{y} \in (0, 1)$. Then $\psi_2(\bar{y}) = \psi_1(\bar{y})$ and, since $\psi_2 \geq \psi_1$ in $(0, 1)$, then \bar{y} is a point of minimum for $\psi_2 - \psi_1$, so that $(\psi_2 - \psi_1)'(\bar{y}) = 0$. We infer from (3.4) that

$$0 = (\psi_2 - \psi_1)'(\bar{y}) + \frac{p}{p-1} \left[\psi_2^{(p-1)/p}(\bar{y}) - \psi_1^{(p-1)/p}(\bar{y}) \right] = \frac{p}{p-1}(a_2^{2-p} - a_1^{2-p})(1 - \bar{y}),$$

which leads to $a_1 = a_2$, hence a contradiction. This proves that $\bar{y} = 1$ and thereby completes the proof of Lemma 3.2. \square

3.2. Behavior of $\psi(y, a)$ as $y \rightarrow 1$. We next describe the shape of $\psi(\cdot, a)$.

Lemma 3.3. *Given $a > 0$ there is $y_a \in (0, 1)$ such that*

$$\psi'(y_a, a) = 0, \quad \psi'(y, a)(y - y_a) < 0, \quad y \in (0, 1) \setminus \{y_a\}. \quad (3.8)$$

Moreover there is $\ell(a) \geq 0$ such that

$$\lim_{y \rightarrow 1} \psi(y, a) = \ell(a), \quad (3.9)$$

and

$$\psi(y, a) \geq a^{p(2-p)/(p-1)}(1 - y)^{p/(p-1)}, \quad y \in (y_a, 1). \quad (3.10)$$

Proof. We define $y_a := \inf\{y \in (0, 1) : \psi'(y) = 0\}$ and note that $y_a > 0$ by (3.6). Assume for contradiction that $y_a = 1$. Then $\psi' > 0$ in $[0, 1)$ and it follows from (3.1) and (3.4) that

$$0 \leq \psi(y)^{(p-1)/p} \leq a^{2-p}(1 - y), \quad y \in (0, 1).$$

Consequently, $\psi(1) = 0 = \psi(0)$ which contradicts the strict monotonicity of ψ . Therefore $y_a \in (0, 1)$ with $\psi' > 0$ in $[0, y_a)$, $\psi'(y_a) = 0$, and

$$\psi''(y_a) = -\psi(y_a)^{-1/p}\psi'(y_a) - \frac{pa^{2-p}}{p-1} = -\frac{pa^{2-p}}{p-1} < 0.$$

In particular, ψ' is negative in a right neighborhood of y_a . Assume for contradiction that there is $z \in (y_a, 1)$ such that $\psi'(y) < 0$ for $y \in (y_a, z)$ and $\psi'(z) = 0$. Then $\psi''(z) \geq 0$, while (3.4) entails that $\psi''(z) = -pa^{2-p}/(p-1) < 0$, and a contradiction. We have thus proved (3.8) which, together with (3.1), implies in particular that ψ is positive and decreasing on $(y_a, 1)$, hence (3.9).

Finally, if $y \in [y_a, 1)$, one has $\psi'(y) < 0$ by (3.8) and we infer from (3.4) that

$$\frac{p}{p-1}\psi(y)^{(p-1)/p} \geq \frac{pa^{2-p}}{p-1}(1 - y),$$

from which (3.10) readily follows. \square

The next step, which is the cornerstone of the classification of the behavior of $\psi(\cdot, a)$ according to the value of a , is to elucidate the behavior of $\psi(y, a)$ as $y \rightarrow 1$. While it is obvious if $\ell(a) > 0$, more information is needed when $\ell(a) = 0$.

Lemma 3.4. *Let $a > 0$ and assume that $\ell(a) = 0$. Then $y \mapsto \psi(y, a)(1 - y)^{-p}$ has a limit as $y \rightarrow 1$ and*

$$0 \leq \psi(y, a) \leq \kappa(1 - y)^p, \quad y \in (0, 1), \quad (3.11)$$

$$\lim_{y \rightarrow 1} \psi(y, a)(1 - y)^{-p} \in \{0, \kappa\}, \quad (3.12)$$

where $\kappa := (p - 1)^{-p}$.

Proof. It readily follows from (3.4) that, for $y \in (0, 1)$,

$$\begin{aligned} (\psi^{1/p})'(y) + \frac{1}{p-1} &= \frac{1}{p} \psi(y)^{-(p-1)/p} \psi'(y) + \frac{1}{p-1} \\ &= \frac{a^{2-p}}{p-1} (1-y) \psi(y)^{-(p-1)/p} \geq 0 . \end{aligned}$$

Integrating the above differential inequality over $(y, 1)$ and using $\ell(a) = 0$ lead us to

$$\frac{1}{p-1} \geq \psi(y)^{1/p} + \frac{y}{p-1} , \quad y \in (0, 1) ,$$

hence (3.11).

We next define

$$\varphi(y) = \varphi(y, a) := \psi(y, a)(1-y)^{-p} , \quad y \in [0, 1) , \quad (3.13)$$

and deduce from (3.4)-(3.5) that φ solves

$$\varphi'(y) + \frac{p}{1-y} \varphi(y)^{(p-1)/p} (\kappa^{1/p} - \varphi(y)^{1/p}) = \frac{pa^{2-p}}{p-1} (1-y)^{1-p} , \quad y \in (0, 1) , \quad (3.14)$$

$$\varphi(0) = 0 . \quad (3.15)$$

Integrating (3.16) over $(0, y)$ and using (3.15) give

$$\varphi(y) + p \int_0^y \Phi(z) dz = \frac{pa^{2-p}}{(p-1)(2-p)} [1 - (1-y)^{2-p}] \quad (3.16)$$

for $y \in [0, 1)$, where

$$\Phi(y) := \varphi(y)^{(p-1)/p} \left[\frac{\kappa^{1/p} - \varphi(y)^{1/p}}{1-y} \right] , \quad y \in [0, 1) .$$

We then infer from (3.11) that $\Phi \geq 0$ in $(0, 1)$, which gives, together with (3.16) and the non-negativity of φ ,

$$0 \leq p \int_0^y \Phi(z) dz \leq \frac{pa^{2-p}}{(p-1)(2-p)} , \quad y \in [0, 1) .$$

Consequently, $\Phi \in L^1(0, 1)$ and (3.16) ensures that $\varphi(y)$ has a limit L as $y \rightarrow 1$ given by

$$\lim_{y \rightarrow 1} \varphi(y) = L := \frac{pa^{2-p}}{(p-1)(2-p)} - p \int_0^1 \Phi(y) dy .$$

Recalling the definition of Φ , we realize that

$$\lim_{y \rightarrow 1} (1-y) \Phi(y) = L^{(p-1)/p} (\kappa^{1/p} - L^{1/p}) ,$$

and the integrability of Φ implies that $L \in \{0, \kappa\}$. □

3.3. Classification. The outcome of Lemma 3.3 and Lemma 3.4 allows us to split the range of a into three sets according to the behavior of $\psi(y, a)$ as $y \rightarrow 1$. More precisely, we define

$$\begin{aligned}\mathcal{A} &:= \{a \in (0, \infty) : \ell(a) > 0\} , \\ \mathcal{B} &:= \left\{ a \in (0, \infty) : \lim_{y \rightarrow 1} \psi(y, a)(1-y)^{-p} = \kappa \right\} , \\ \mathcal{C} &:= \left\{ a \in (0, \infty) : \lim_{y \rightarrow 1} \psi(y, a)(1-y)^{-p} = 0 \right\} .\end{aligned}$$

Indeed, according to Lemma 3.3 and Lemma 3.4, the sets \mathcal{A} , \mathcal{B} , and \mathcal{C} are disjoint and

$$\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} = (0, \infty) .$$

We now provide a more accurate description of these sets and begin with \mathcal{A} .

Lemma 3.5. *There holds*

$$a \in \mathcal{A} \text{ if and only if } \sup_{y \in [0,1)} \{\psi(y, a)(1-y)^{-p}\} > \kappa . \quad (3.17)$$

Furthermore, there is $a^* > 0$ such that $\mathcal{A} = (a^*, \infty)$.

Proof. As in the proof of Lemma 3.4, see Equation (3.13), we set $\varphi(y) = \psi(y)(1-y)^{-p}$ for $y \in [0, 1)$.

Step 1. If $a \in \mathcal{A}$ then $\ell(a) > 0$, from which we readily deduce that $\varphi(y) \rightarrow \infty$ as $y \rightarrow 1$, and obviously $\sup_{y \in [0,1)} \{\varphi(y)\} > \kappa$. Conversely, if $\sup_{y \in [0,1)} \{\varphi(y)\} > \kappa$, then necessarily $\ell(a) \neq 0$ according to Lemma 3.4 and thus $a \in \mathcal{A}$.

Step 2. We claim that \mathcal{A} is non-empty. Indeed, assume for contradiction that $\mathcal{A} = \emptyset$, so that $\ell(a) = 0$ for all $a > 0$. We then infer from (3.11), (3.14), and the non-negativity of φ that

$$\varphi'(y) \geq \frac{pa^{2-p}}{p-1}(1-y)^{1-p} - \frac{p\kappa^{1/p}}{1-y}\varphi(y)^{(p-1)/p} \geq \frac{pa^{2-p}}{p-1}(1-y)^{1-p} - \frac{p\kappa}{1-y}$$

for $y \in (0, 1)$. Integrating over $(0, 1/2)$ and using once more (3.11) give

$$\kappa \geq \varphi(1/2) \geq \frac{pa^{2-p}}{(p-1)(2-p)} (1 - 2^{p-2}) - p\kappa \log 2 ,$$

and a contradiction for a large enough. Consequently, \mathcal{A} is non-empty.

Step 3. We put $a^* := \inf \mathcal{A}$. A straightforward consequence of the characterization (3.17) and the monotonicity of $\psi(\cdot, a)$ with respect to a established in Lemma 3.2 and (3.17) is that $(a^*, \infty) \subset \mathcal{A}$. Furthermore, if $a \in \mathcal{A}$, then $\ell(a) > 0$ and it follows from Lemma 3.2 that, for $\delta \in (0, a)$

$$0 < \ell(a) \leq \ell(a + \delta) \quad \text{and} \quad 0 < \ell(a) \leq \ell(a - \delta) + K(p)\delta^{2-p} .$$

Therefore $\ell(a + \delta) > 0$ and $\ell(a - \delta) > 0$ for δ small enough, so that $(a - \delta, a + \delta) \subset \mathcal{A}$ for $\delta > 0$ small enough. In particular, \mathcal{A} is open and thus coincides with (a^*, ∞) . \square

Concerning \mathcal{C} one has the following result.

Lemma 3.6. *The following statements are equivalent:*

- (c1) $a \in \mathcal{C}$.
- (c2) $\sup_{y \in [0,1]} \{\psi(y, a)(1 - y)^{-p}\} < \kappa$.
- (c3) *The derivative $\varphi'(\cdot, a)$ of the function $\varphi(\cdot, a)$ defined in (3.13) vanishes at least once in $(0, 1)$.*
- (c4) *There is $Y_a \in (0, 1)$ such that*

$$\varphi'(Y_a, a) = 0, \quad \varphi'(y, a)(y - Y_a) < 0, \quad y \in (0, 1) \setminus \{Y_a\}. \quad (3.18)$$

Furthermore there is $a_* > 0$ such that $\mathcal{C} = (0, a_*)$.

Proof. Recall that $\varphi(y) = \psi(y)(1 - y)^{-p}$ for $y \in [0, 1)$, see Equation (3.13).

Step 1. Assume first that $\sup_{y \in [0,1]} \{\varphi(y)\} < \kappa$. This property readily implies that $\ell(a) = 0$ and we deduce from Lemma 3.4 that the limit of $\varphi(y)$ as $y \rightarrow 1$ is necessarily zero. Therefore $a \in \mathcal{C}$ and we have proved that (c2) \Rightarrow (c1).

Consider now $a \in \mathcal{C}$. Since $\varphi(0) = 0$ by (3.15) and $\varphi(y) \rightarrow 0$ as $y \rightarrow 1$, a generalization of Rolle's theorem guarantees that φ' vanishes at least once in $(0, 1)$, and (c1) \Rightarrow (c3).

Assume next that φ' vanishes at least once in $(0, 1)$ and denote its smallest zero by $Y_a \in (0, 1)$. Since $\varphi'(0) = pa^{2-p}/(p-1) > 0$ by (3.14), the function φ' is positive in $[0, Y_a)$ and it follows from (3.14) that

$$\varphi''(Y_a) = -\frac{p(2-p)a^{2-p}}{p-1}(1 - Y_a)^{-p} < 0.$$

Consequently, φ' is negative in a right neighborhood of Y_a . Assume for contradiction that there is $Y_1 \in (Y_a, 1)$ such that $\varphi'(y) < 0$ for $y \in (Y_a, Y_1)$ and $\varphi'(Y_1) = 0$. Then $\varphi''(Y_1) \geq 0$ while (3.14) implies that $\varphi''(Y_1) = -p(2-p)a^{2-p}(1 - Y_1)^{-p}/(p-1) < 0$, and a contradiction. Therefore $\varphi' < 0$ in $(Y_a, 1)$ and we have shown that φ enjoys the property (3.18), that is, (c3) \Rightarrow (c4).

Finally, assume that φ satisfies (3.18). Then $\sup_{y \in [0,1]} \{\varphi(y)\} = \varphi(Y_a)$ and we deduce from (3.14) that

$$\frac{p}{1 - Y_a} \varphi(Y_a)^{(p-1)/p} [\kappa^{1/p} - \varphi(Y_a)^{1/p}] = \frac{pa^{2-p}}{p-1} (1 - Y_a)^{1-p} > 0.$$

Consequently $\varphi(Y_a) < \kappa$ and (c4) \Rightarrow (c2).

Step 2. We now check that \mathcal{C} is non-empty. To this end, consider $a > 0$ such that

$$a^{2-p} \leq (p-1)^{p-1}/p^p = \max_{A \in (0,1)} \{A^{(p-1)/p} - A\}.$$

We fix $A \in (0, 1)$ such that $A^{(p-1)/p} - A \geq a^{2-p}$ and set $\Sigma_A(y) = A(1-y)^{p/(p-1)}$ for $y \in [0, 1)$. On the one hand,

$$\begin{aligned} \Sigma'_A(y) + \frac{p}{p-1} \Sigma_A(y)^{(p-1)/p} &= \frac{p}{p-1} (1-y) [A^{(p-1)/p} - A(1-y)^{(2-p)/(p-1)}] \\ &\geq \frac{p}{p-1} (1-y) [A^{(p-1)/p} - A] \\ &\geq \frac{p}{p-1} (1-y) a^{2-p} = \psi'(y) + \frac{p}{p-1} \psi(y)^{(p-1)/p} \end{aligned}$$

for $y \in (0, 1)$. On the other hand, $\Sigma_A(0) = A > 0 = \psi(0)$. We are then in a position to apply Lemma 3.1 with $(\xi_1, \xi_2) = (\psi, \Sigma_A)$ to conclude that

$$0 \leq \psi(y) \leq A(1-y)^{p/(p-1)}, \quad y \in [0, 1).$$

Since $p < p/(p-1)$, the above estimate ensures that $a \in \mathcal{C}$ and we have thus shown that \mathcal{C} is non-empty and contains the interval $(0, (p-1)^{(p-1)/(2-p)} p^{-p/(2-p)})$.

Step 3. Introducing $a_* := \sup\{\mathcal{C}\} > 0$, we infer from the monotonicity of $\psi(\cdot, a)$ with respect to a (Lemma 3.2) that $(0, a_*) \subset \mathcal{C}$.

Assume for contradiction that $a_* \in \mathcal{C}$. Owing to (3.18) there are $\delta > 0$ and $\varepsilon > 0$ such that

$$\varphi'(Y_{a_*} + \delta, a_*) < -2\varepsilon < \varphi'(Y_{a_*}, a_*) = 0 < 2\varepsilon < \varphi'(Y_{a_*} - \delta, a_*) \quad (3.19)$$

and

$$\frac{Y_{a_*}}{2} \leq Y_{a_*} - \delta < Y_{a_*} < Y_{a_*} + \delta \leq \frac{1 + Y_{a_*}}{2}. \quad (3.20)$$

Thanks to Lemma 3.2, $\varphi'(\cdot, a)$ depends continuously on a on $[0, (1 + Y_{a_*})/2]$ and we infer from (3.19) and (3.20) that there is $\alpha > 0$ small enough such that

$$\varphi'(Y_{a_*} + \delta, a) < -\varepsilon < \varepsilon < \varphi'(Y_{a_*} - \delta, a), \quad a \in [a_* - \alpha, a_* + \alpha].$$

In particular, for all $a \in [a_* - \alpha, a_* + \alpha]$, the function $\varphi'(\cdot, a)$ has a zero inside the interval $(Y_{a_*} - \delta, Y_{a_*} + \delta)$. According to (c3), this means that $[a_* - \alpha, a_* + \alpha] \subset \mathcal{C}$, which contradicts the definition of a_* . Therefore $a_* \notin \mathcal{C}$ and $\mathcal{C} = (0, a_*)$. \square

We finally turn to the description of the set \mathcal{B} and show that it is a singleton.

Proposition 3.7. *There holds $a_* = a^*$ and $\mathcal{B} = \{a_*\}$, where a^* and a_* are defined in Lemma 3.5 and Lemma 3.6, respectively.*

Proof. Owing to Lemma 3.5 and Lemma 3.6 there holds $\mathcal{B} = [a_*, a^*]$ and $\varphi'(\cdot, a) > 0$ in $(0, 1)$ for $a \in \mathcal{B}$, recalling that the function $\varphi(\cdot, a)$ is defined by (3.13). Introducing $G := \varphi(\cdot, a^*) - \varphi(\cdot, a_*)$ it follows from Lemma 3.2 and (3.14) that $G \geq 0$ and

$$\begin{aligned} G'(y) + \frac{pk^{1/p}}{1-y} [\varphi(y, a^*)^{(p-1)/p} - \varphi(y, a_*)^{(p-1)/p}] \\ = p \frac{G(y)}{1-y} + \frac{p}{p-1} [(a^*)^{2-p} - (a_*)^{2-p}] (1-y)^{1-p} \end{aligned} \quad (3.21)$$

for $y \in (0, 1)$. Since $a_* \in \mathcal{B}$ we deduce from the definition of \mathcal{B} that there is $Y \in (0, 1)$ such that

$$\varphi(y, a_*) \geq \left(p - \frac{1}{2}\right)^{-p}, \quad y \in [Y, 1).$$

Therefore, for $y \in [Y, 1)$,

$$\begin{aligned} \varphi(y, a^*)^{(p-1)/p} - \varphi(y, a_*)^{(p-1)/p} &= \frac{p-1}{p} \int_{\varphi(y, a_*)}^{\varphi(y, a^*)} z^{-1/p} dz \\ &\leq \frac{p-1}{p} \varphi(y, a_*)^{-1/p} G(y) \\ &\leq \frac{(p-1)(2p-1)}{2p} G(y). \end{aligned}$$

Combining the above estimate with (3.21) gives, for $y \in [Y, 1)$,

$$\begin{aligned} G'(y) + \frac{p}{(p-1)(1-y)} \frac{(p-1)(2p-1)}{2p} G(y) &\geq \frac{pG(y)}{1-y} + \frac{p}{p-1} [(a^*)^{2-p} - (a_*)^{2-p}] (1-y)^{1-p} \\ &\geq \frac{pG(y)}{1-y}, \end{aligned}$$

whence, after easy manipulations,

$$G'(y) \geq \frac{G(y)}{2(1-y)}, \quad y \in [Y, 1).$$

Integrating the above differential inequality on $[Y, y]$ for some $y \in (Y, 1)$, we find

$$G(y) \geq G(Y) \sqrt{\frac{1-Y}{1-y}}, \quad y \in (Y, 1). \quad (3.22)$$

Assume now for contradiction that $a^* > a_*$. We deduce from Lemma 3.2 and the fact that $Y \in (0, 1)$ that $\varphi(Y, a^*) > \varphi(Y, a_*)$, that is, $G(Y) > 0$. It then follows from (3.22) that $G(y) \rightarrow \infty$ as $y \rightarrow 1$. However, the definition of \mathcal{B} entails that $G(y) \rightarrow 0$ as $y \rightarrow 1$, clearly in contradiction with the previous assertion. Therefore $a_* = a^*$ and the proof of Proposition 3.7 is complete. \square

3.4. Refined asymptotics as $y \rightarrow 1$ for $a \in \mathcal{C}$. The final step is to identify the behavior of $\psi(y, a)$ as $y \rightarrow 1$ for $a \in \mathcal{C}$.

Lemma 3.8. *If $a \in \mathcal{C}$ then*

$$\lim_{y \rightarrow 1} \psi(y, a)(1-y)^{-p/(p-1)} = a^{p(2-p)/(p-1)}.$$

Proof. Let $a \in \mathcal{C}$.

Step 1. We first prove that there exists $M > a^{p(2-p)/(p-1)}$ such that

$$\psi(y) \leq M(1-y)^{p/(p-1)}, \quad y \in [0, 1). \quad (3.23)$$

Indeed, let $\varepsilon \in (0, 1)$ to be determined later and define

$$\sigma_\varepsilon(y) := \frac{1}{2\varepsilon^{p(2-p)/(p-1)}}(1-y)^{p/(p-1)}, \quad y \in (0, 1).$$

Owing to the definition of \mathcal{C} , there is $\bar{\varepsilon} \in (0, 1)$ such that $\psi(y) \leq (1-y)^p/2$ for $y \in (1-\bar{\varepsilon}, 1)$. On the one hand, if $\varepsilon \in (0, \bar{\varepsilon})$, there holds

$$\sigma_\varepsilon(1-\varepsilon) = \frac{\varepsilon^p}{2} \geq \psi(1-\varepsilon).$$

On the other hand, for $y \in (1-\varepsilon, 1)$,

$$\begin{aligned} \sigma'_\varepsilon(y) + \frac{p}{p-1}\sigma_\varepsilon(y)^{(p-1)/p} &= \frac{p}{p-1}(1-y) \left[\frac{1}{2^{(p-1)/p}\varepsilon^{2-p}} - \frac{(1-y)^{(2-p)/(p-1)}}{2\varepsilon^{p(2-p)/(p-1)}} \right] \\ &\geq \frac{p}{p-1}(1-y) \frac{2^{1/p}-1}{2\varepsilon^{2-p}} \\ &\geq \frac{pa^{2-p}}{p-1}(1-y) = \psi'(y) + \frac{p}{p-1}\psi(y)^{(p-1)/p}, \end{aligned}$$

as soon as

$$\frac{2^{1/p}-1}{2\varepsilon^{2-p}} \geq a^{2-p}. \quad (3.24)$$

We next choose $\varepsilon \in (0, \bar{\varepsilon})$ satisfying (3.24). This allows us to apply Lemma 3.1 with $(\xi_1, \xi_2) = (\psi, \sigma_\varepsilon)$ in order to obtain that $\psi(y) \leq \sigma_\varepsilon(y)$ for $y \in (0, 1-\varepsilon)$. This inequality extends to the whole interval $(0, 1)$, possibly taking a smaller value of ε .

Step 2. The goal of this step is to improve (3.23). To this end, fix $A \in (a^{p(2-p)/(p-1)}, M)$ and $\varepsilon \in (0, 1)$ such that

$$\varepsilon^{(2-p)/(p-1)} < \frac{A^{(p-1)/p} - a^{2-p}}{2M}. \quad (3.25)$$

We define

$$\tau(y) := \left(A + \frac{M-A}{\varepsilon}(1-y) \right) (1-y)^{p/(p-1)}, \quad y \in (0, 1),$$

and deduce from (3.23) that

$$\tau(1-\varepsilon) = M\varepsilon^{p/(p-1)} \geq \psi(1-\varepsilon).$$

In addition, we infer from (3.4) and (3.25) that, for $y \in (1 - \varepsilon, 1)$,

$$\begin{aligned}
& \tau'(y) + \frac{p}{p-1} \tau(y)^{(p-1)/p} \\
& \geq \frac{p}{p-1} (1-y) \left[A^{(p-1)/p} - A(1-y)^{(2-p)/(p-1)} - \frac{2p-1}{p} \frac{M-A}{\varepsilon} (1-y)^{1/(p-1)} \right] \\
& \geq \frac{p}{p-1} (1-y) \left[A^{(p-1)/p} - \left(A + \frac{2p-1}{p} (M-A) \right) \varepsilon^{(2-p)/(p-1)} \right] \\
& \geq \frac{p}{p-1} (1-y) \left[A^{(p-1)/p} - 2M \varepsilon^{(2-p)/(p-1)} \right] \\
& \geq \frac{pa^{2-p}}{p-1} (1-y) = \psi'(y) + \frac{p}{p-1} \psi(y)^{(p-1)/p} .
\end{aligned}$$

Applying Lemma 3.1 with $(\xi_1, \xi_2) = (\psi, \tau)$ implies that $\psi(y) \leq \tau(y)$ for $y \in (1 - \varepsilon, 1)$. Consequently,

$$\frac{\psi(y)}{(1-y)^{p/(p-1)}} \leq A + \frac{M-A}{\varepsilon} (1-y) , \quad y \in (1 - \varepsilon, 1) ,$$

from which we deduce that

$$\limsup_{y \rightarrow 1} \frac{\psi(y)}{(1-y)^{p/(p-1)}} \leq A .$$

As A is arbitrarily chosen in $(a^{p(2-p)/(p-1)}, M)$, we end up with

$$\limsup_{y \rightarrow 1} \frac{\psi(y)}{(1-y)^{p/(p-1)}} \leq a^{p(2-p)/(p-1)} .$$

Since

$$\liminf_{y \rightarrow 1} \frac{\psi(y)}{(1-y)^{p/(p-1)}} \geq a^{p(2-p)/(p-1)}$$

by (3.10), the claimed result follows. \square

4. PROOF OF THEOREM 1.1

We now undo the transformation (3.1) and interpret the outcome of Section 3 in terms of $f(\cdot, a)$. Let $a \in (0, \infty)$. It follows from (2.1) and (3.2) that

$$f'(r) = -a\psi \left(1 - \frac{f(r)}{a} \right)^{1/p} , \quad r \in [0, R(a)) . \quad (4.1)$$

Since $\psi(y) \sim pa^{2-p}y/(p-1)$ as $y \rightarrow 0$ and $p > 1$, the function $z \mapsto \psi(1-z)^{-1/p}$ defined on $(0, 1)$ belongs to $L^1(z_0, 1)$ for all $z_0 > 0$. We may thus integrate (4.1) and find

$$\int_{f(r)/a}^1 \frac{dz}{\psi(1-z)^{1/p}} = r , \quad r \in [0, R(a)) . \quad (4.2)$$

Case 1: $a \in \mathcal{A}$. According to the definition of \mathcal{A} , $\psi(y)$ has a positive limit $\ell(a) > 0$ as $y \rightarrow 1$ and the function $z \mapsto \psi(1-z)^{-1/p}$ actually belongs to $L^1(0, 1)$. We then deduce from (4.2) that

$$\int_0^1 \frac{dz}{\psi(1-z)^{1/p}} = R(a) ,$$

that is, $R(a) < \infty$. Furthermore, $f'(R(a)) = -a\ell(a)^{1/p} < 0$ by (4.1) and the proof of Theorem 1.1 (a) is complete.

Case 2: $a \in \mathcal{B}$. By Proposition 3.7 there holds $a = a_*$ and the definition of \mathcal{B} ensures that $\psi(1-z)^{1/p} \sim z/(p-1)$ as $z \rightarrow 0$. Therefore $z \mapsto \psi(1-z)^{-1/p}$ does not belong to $L^1(0, 1)$ and we infer from (4.2) that $R(a_*) = \infty$ and

$$r \sim -(p-1) \log(f(r)) \quad \text{as } r \rightarrow \infty .$$

In particular, there is $R > 0$ such that

$$-\frac{p-1}{r} \log(f(r)) \geq 1 - \frac{2-p}{2} = \frac{p}{2} , \quad r \geq R ,$$

from which we deduce that

$$\int_R^\infty e^r f(r) dr \leq \int_R^\infty e^{-(2-p)r/2(p-1)} dr < \infty , \quad (4.3)$$

since $p \in (1, 2)$. Recalling (2.1), it follows from (2.2) after integration that

$$-e^r |f'(r)|^{p-1} = - \int_0^r e^\sigma f(\sigma) d\sigma ,$$

which, together with (4.3), guarantees that $e^r |f'(r)|^{p-1}$ has a finite limit as $r \rightarrow \infty$ and

$$\lim_{r \rightarrow \infty} e^r |f'(r)|^{p-1} = I := \int_0^\infty e^r f(r) dr .$$

We then infer from the above property, (4.1), and the behavior of $\psi(y)$ as $y \rightarrow 1$ that

$$f'(r) \sim -I^{1/(p-1)} e^{-r/(p-1)} \quad \text{and} \quad f(r) \sim -\frac{f(r)}{p-1} \quad \text{as } r \rightarrow \infty ,$$

so that $f(r) \sim (p-1)I^{1/(p-1)} e^{-r/(p-1)}$ as $r \rightarrow \infty$. We have thus proved Theorem 1.1 (b).

Case 3: $a \in \mathcal{C}$. In that case, $\psi(1-z)^{1/p} \sim a^{(2-p)/(p-1)} z^{1/(p-1)}$ as $z \rightarrow 0$ by Lemma 3.8. Since $p \in (1, 2)$ the function $z \mapsto \psi(1-z)^{-1/p}$ does not belong to $L^1(0, 1)$ and we infer from (4.2) that $R(a) = \infty$ and

$$\frac{p-1}{2-p} \left(\frac{f(r)}{a} \right)^{-(2-p)/(p-1)} \sim a^{(2-p)/(p-1)} r \quad \text{as } r \rightarrow \infty ,$$

hence Theorem 1.1 (c).

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